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Linear theory of the circulation of a stratified ocean

By JOSEPH PEDLOSKY
Department of the Geophysical Sciences, University of Chicago

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A linear model of the circulation of a stratified ocean, in a closed basin, driven by both wind stress and heating is presented. Particular attention is given to the interdependence of the primary features of the oceanic circulation. The upwelling process is studied in detail and it is shown that the complete determination of the mid-ocean thermocline solution depends on the upwelling in the boundary layers on the ocean basin's side walls. The morphology of the side wall boundary layers as a function of the stratification is also discussed.

1. Introduction

One of the central problems in dynamical oceanography is understanding how surface heating and wind stress drive the steady-state general oceanic circulation. The theoretical description of the observed oceanic density structure with its steep vertical gradient (the thermocline) and associated currents is one of the chief goals. The problem presents formidable difficulties; consequently, most of the theoretical investigations have dealt, in one way or another, with highly idealized models. Indeed, the present study is no exception.

In the past, the construction of models has generally proceeded along two paths, which loosely correspond to either studies of the wind-driven circulation or 'thermocline' studies which concentrate on deducing from a given surface-temperature (and wind-stress) distribution the mid-ocean density and current structure.

In investigations of the thermocline problem (e.g. Robinson & Stommel 1959; Robinson & Welander 1963) the model dynamics was assumed to be linear, i.e. the pressure gradient was balanced in the horizontal by the Coriolis forces (geostrophic approximation) and in the vertical by buoyancy forces (hydrostatic approximation). The heat equation for the temperature (or density), however, was non-linear since both advective and diffusive processes are important. This non-linearity hampers the analytical treatment of the problem and progress has been made only for open ocean basins, usually bounded by a single meridional coast. In a sense the 'geometry' is, in such treatments, sacrificed for greater realism in the 'dynamics'. However, there are instances of rotating, stratified fluids in which the lateral, vertical boundaries are determining factors in the total circulation and therefore of crucial importance (e.g. Barcilon & Pedlosky 1967).

In contrast, noting that over most of the ocean the vertically integrated horizontal velocity (the transport) is independent of the density structure, investigations of the wind-driven circulation problem have usually ignored the effects of stratification while recognizing the importance of the ocean basin's boundedness. These investigations (e.g. Stommel 1948; Munk 1950) have been successful in explaining the observed western intensification of the oceanic circulation as an effect of the lateral boundaries on a spherical globe, but the resulting complexities due to geometry are relieved by ignoring the ocean's stratification.

A different point of view is taken in this paper in an attempt to illuminate certain questions left unanswered by the previous approaches. Namely what is the effect of lateral boundaries on the thermocline-associated circulation and the effect of continuous stratification on the results of the wind-driven transport theories? The goal here is to formulate a well-posed problem for the combined wind-driven and thermally driven ocean circulation in an enclosed basin of finite depth exhibiting the most interesting features of the oceanic circulation, which is at the same time, analytically, sufficiently tractable so that it will be clear in what way the various elements of the problem, hitherto studied in isolation, are interdependent. To fulfil this last requirement all the equations of motion, including the heat equation, are linearized. The heat-advection process is taken into account by linearizing about a given vertical stratification. The shortcomings of this approach, from an oceanographic point of view, are obvious in so far as the necessary physical model for the linearization departs markedly from the real physical situation. Nevertheless, all the fundamental physical processes are still represented; and together with turbulent eddy coefficients of viscosity and thermal diffusivity a tractable, well-posed boundary-value problem can be pursued. I stress again that the aim of this study is to explore the interdependence of certain fundamental oceanographic phenomena in this *simple* model, which in the past have perforce been studied separately owing to the greater degree of complexity which arises in more physical realistic models. It is found, for example, that the structure of certain boundary-layer regions and their morphology as a function of the stratification depends on the nature of the interior and vice versa; further, the complete specification of the interior, mid-ocean circulation is only possible when the boundary-layer problem of upwelling at the coast has been solved. This latter result was anticipated in an earlier study (Pedlosky 1968) of the wind-driven circulation of a homogeneous ocean; it is even more forcefully true when the ocean is stratified. Some of these features would be missed in a partial treatment of the problem which did not explicitly recognize the necessity of closing the circulation by satisfying all the necessary boundary conditions.

2. The model

Consider a rectangular ocean basin of constant depth D bounded by rigid, thermally insulating walls on $x = 0, L$ and $y = 0, bL$, where x, y and z are co-ordinates measuring eastward, northward and upward respectively ($0 \leqslant z \leqslant D$).

The effect of the earth's sphericity is modelled by using a variable Coriolis parameter f which is a linear function of y, i.e.

$$f = f_0 + \beta_* y.$$

This is the traditional β-plane approximation first introduced (in a meteorological context) by Rossby (1939) and is a suitable idealization of the dynamical effects of sphericity in middle latitudes. Cartesian co-ordinates are then used for an otherwise planar system. The imposed temperature variations are sufficiently small so that a Boussinesq approximation is presumed valid throughout with a linear state relation between temperature and density. The effect of salinity in separately affecting the density field is ignored. The equations of steady motion are then:

$$uu_x + vu_y + wu_z - fv = -p_x + \nu_H(u_{xx} + u_{yy}) + \nu_V u_{zz},$$

$$uv_x + vv_y + wv_z + fu = p_y + \nu_H(v_{xx} + v_{yy}) + \nu_V v_{zz},$$

$$uw_x + vw_y + ww_z = -p_z - \rho g + \nu_H(w_{xx} + w_{yy}) + \nu_V w_{zz},$$

$$u_x + v_y + w_z = 0,$$

$$uT_x + vT_y + wT_z = \kappa_H(T_{xx} + T_{yy}) + \kappa_V T_{zz},$$

$$\rho = \rho_0(1 - \alpha(T - T_0)).$$

(u, v, w) are the eastward, northward and upward components of velocity. The (variable) rotation vector $\frac{1}{2}f$ is anti-parallel to the direction of gravity, g. p, ρ and T are the pressure, density and temperature respectively. The density and temperature are related linearly by the last equation wherein ρ_0 and T_0 are reference levels for ρ and T while α is the coefficient of thermal expansion of the fluid. Finally, eddy coefficients, meant to parameterize the effects of small-scale turbulent diffusion of momentum and heat, are introduced. These are ν_H and κ_H for the horizontal diffusion of momentum and heat, while ν_V and κ_V are meant to represent the process of vertical diffusion of momentum and heat. They are taken in this model as constants. Although the fluid is stratified it is incompressible.

The temperature field is partitioned in the following way:

$$T = T_0 + (\Delta T_V)(z/D) + (\Delta T_H)T'(x, y, z).$$

The constants (ΔT_V) and (ΔT_H) measure, respectively, the size of the basic stable stratification and the horizontally variable temperature imposed on the ocean's upper surface at $z = D$. In order to effect a linearization of the problem it is assumed that $(\Delta T_V) \gg (\Delta T_H)$. In short, the mean ocean thermal structure has been highly simplified for the purpose of mathematical simplicity.

In the real ocean the mean (and averaged) temperature gradient decays with depth and is maintained by the motion itself. In this model this feature holds true only for the small, motion-produced anomaly, T'.

Similarly, a wind-stress $\boldsymbol{\tau}(x, y) = \tau_0 \boldsymbol{\tau}'(x, y)$ is specified at $z = D$, τ_0 being a measure of the stress amplitude.

Non-dimensional variables, denoted by primes, are introduced as follows:

$$(x,y) = L(x',y'), \quad z = Dz',$$

$$(u,v) = \alpha(\Delta T_H)\frac{gD}{f_0 L}(u',v'), \quad w = \alpha(\Delta T_H)\frac{gD^2}{f_0 L^2}w',$$

$$p = -\rho_0 gDz' + \tfrac{1}{2}\rho_0\alpha g(\Delta T_V)z'^2 D + \rho_0\alpha(\Delta T_H)gDp',$$

$$f = f_0\left(1+\frac{\beta_* L}{f_0}y'\right) = f_0(1+\beta y') = f_0 f'.$$

The equations of motion become, after eliminating the density with the use of the state equation, and then dropping the prime notation for dimensionless variables:

$$\epsilon_T(uu_x+vu_y+wu_z)-fv = -p_x+\tfrac{1}{2}E_H(u_{xx}+u_{yy})+\tfrac{1}{2}E_V u_{zz}, \quad (2.1a)$$

$$\epsilon_T(uv_x+vv_y+wv_z)+fu = -p_y+\tfrac{1}{2}E_H(v_{xx}+v_{yy})+\tfrac{1}{2}E_V v_{zz}, \quad (2.1b)$$

$$\delta^2\epsilon_T(uw_x+vw_y+ww_z) = -p_z+T+\delta^2[\tfrac{1}{2}E_H(w_{xx}+w_{yy})+\tfrac{1}{2}E_V w_{zz}], \quad (2.1c)$$

$$\epsilon_T(uT_x+vT_y+wT_z)+wS = \tfrac{1}{2}(E_H/\sigma_H)(T_{xx}+T_{yy})+\tfrac{1}{2}(E_V/\sigma_V)T_{zz}, \quad (2.1d)$$

$$u_x+v_y+w_z = 0. \quad (2.1e)$$

The following dimensionless parameters have emerged:

$\epsilon_T = [\alpha g(\Delta T_H)D]/(f_0^2 L^2)$, the 'thermal' Rossby number;

$E_V = (2\nu_V)/(f_0 D^2)$, the 'vertical' Ekman number;

$E_H = (2\nu_H)/(f_0 L^2)$, the 'horizontal' Ekman number;

$\delta = D/L$, the aspect ratio;

$S = [\alpha g(\Delta T_V)D]/(f_0^2 L^2)$, the stratification number,

and $\sigma_H = \nu_H/\kappa_H$, $\sigma_V = \nu_V/\kappa_V$, the 'horizontal' and 'vertical' turbulent Prandtl numbers.

The only input of energy to drive the circulation occurs at the upper surface of the ocean where the wind stress and temperature, $\mathscr{T}(x,y)$ are specified. All other surfaces are insulated to further heating and allow no slip. Thus,

$$(u,v,w) = 0 \quad \text{on} \quad x = 0,1,\ y = 0,\ b,\ z = 0; \quad (2.2a)$$

$$T_x = 0 \quad \text{on} \quad x = 0,1; \quad (2.2b)$$

$$T_y = 0 \quad \text{on} \quad y = 0,b; \quad (2.2c)$$

$$T_z = 0 \quad \text{on} \quad z = 0; \quad (2.2d)$$

while on $z = 1$,

$$T = \mathscr{T}(x,y); \quad (2.2e)$$

(without any loss of generality it is assumed that

$$\int_0^b dy\int_0^1 dx\,\mathscr{T} = 0)$$

while continuity of stress on $z = 1$ yields

$$\left.\begin{aligned} u_z &= (\epsilon_W/\epsilon_T)E_V^{-\frac{1}{2}}\tau^{(x)}, \\ v_z &= (\epsilon_W/\epsilon_T)E_V^{-\frac{1}{2}}\tau^{(y)}. \end{aligned}\right\} \tag{2.2f}$$

$\tau^{(x)}$ and $\tau^{(y)}$ are the x and y components of $\boldsymbol{\tau}$ while ϵ_W is a Rossby number based on the wind stress, i.e.

$$\epsilon_W = \frac{\tau_0}{\rho_0 f_0 L}\left(\nu_V \frac{f_0}{2}\right)^{-\frac{1}{2}}.$$

Further, variations of the upper surface will be neglected so that

$$w = 0 \quad \text{on} \quad z = 1. \tag{2.2g}$$

The analysis will proceed by neglecting henceforth the terms in the equations of motion proportional to ϵ_T and only the resulting linear system will be considered.

For ease in presentation only, it will arbitrarily be assumed that $E_V = E_H \equiv E$, while $\sigma_V = \sigma_H \equiv \sigma$. No favoured note for either horizontal or vertical mixing is then assumed *a priori*.

Finally, since for the oceanographically relevant parameter range E and δ are small, it is to be expected that boundary layers will be present whose structure will depend on the relative sizes of E, δ and σS. It is realistic to assume that σS is small. Nevertheless, some freedom will be allowed in the magnitude of σS to illustrate the dependence of the circulation on the strength of the stratification. Most of the detailed calculations will be done, however, for the case $E^{\frac{1}{3}} \ll \sigma S \ll 1$.

3. The upper Ekman layer

As is well known, the coupling of the oceanic interior to the surface wind stress, allowing (2.2f) to be satisfied, is accomplished by a viscous boundary layer (the Ekman layer) within a region of $O(E^{\frac{1}{2}})$ of the upper surface.

Within this region the dynamical fields can be represented as follows

$$u = u_I(x, y, z) + u_E(x, y, \hat{z}), \tag{3.1a}$$

$$v = v_I(x, y, z) + v_E(x, y, \hat{z}), \tag{3.1b}$$

$$w = w_I(x, y, z) + E^{\frac{1}{2}} w_E(x, y, \hat{z}), \tag{3.1c}$$

$$T = T_I(x, y, z) + \sigma S E^{\frac{1}{2}} T_E(x, y, \hat{z}), \tag{3.1d}$$

$$p = p_I(x, y, z) + \sigma S E p_E(x, y, \hat{z}), \tag{3.1e}$$

where the I subscripted variables represent the fields below the Ekman layer while the E subscripted variables represent the corrections required within the Ekman layer to satisfy (2.2f), and go to zero as

$$\hat{z} = (1-z)E^{-\frac{1}{2}}$$

becomes large.

It is easy to show that the Ekman layer equations for u_E and v_E are, to lowest order, identical to those for a homogeneous fluid and therefore it can be shown that

$$u_E = (\epsilon_W/2\epsilon_T)f^{-\frac{1}{2}}\exp(-\hat{z}f^{\frac{1}{2}})\,[(\tau^{(y)}-\tau^{(x)})\sin\hat{z}f^{\frac{1}{2}} + (\tau^{(y)}+\tau^{(x)})\cos\hat{z}f^{\frac{1}{2}}], \tag{3.2a}$$

$$v_E = (\epsilon_W/2\epsilon_T)f^{-\frac{1}{2}}\exp(-\hat{z}f^{\frac{1}{2}})\,[(\tau^{(y)}-\tau^{(x)})\cos\hat{z}f^{\frac{1}{2}} - (\tau^{(y)}+\tau^{(x)})\sin\hat{z}f^{\frac{1}{2}}]. \tag{3.2b}$$

With the use of the continuity equation and the condition that w is zero on $z = 1$ we find that

$$w_I(x, y, 1) = \frac{E^{\frac{1}{2}}}{2}\frac{\epsilon_W}{\epsilon_T}\hat{k}\operatorname{curl}\boldsymbol{\tau}/f, \tag{3.3}$$

($\hat{k}$ is a unit vector parallel to the z axis) while, since the temperature correction in the Ekman layer is small $O(\sigma S E^{\frac{1}{2}})$, it is clear that

$$T_I(x, y, 1) = \mathscr{T}(x, y). \tag{3.4}$$

Equations (3.3) and (3.4) will then serve as boundary conditions for the interior flow beneath the Ekman layer. It is clear at this point that two independent problems are being done at once for economy of exposition. The wind-stress driven problem and the thermally driven problem can be done independently because the dynamics has been linearized and the relative strength of the former to the latter is simply proportional to the ratio ϵ_W/ϵ_T. Rather than postulate a formal ordering relationship of this ratio in terms of the other (small) parameters in the problem it is best to treat ϵ_W/ϵ_T as a *trace* constant for that part of the solution forced by the wind stress. Ordering is then done within each problem.

In any event, it will be observed in the next section that u_I and v_I are, in this sense, always of smaller magnitude than u_E and v_E and therefore the horizontal flux of fluid in the Ekman layer $\mathbf{U}_E$, due to the wind stress, is

$$\mathbf{U}_E = [U_E\hat{\imath} + V_E\hat{\jmath}] = E^{\frac{1}{2}}\int_0^\infty (u_E\hat{\imath} + v_E\hat{\jmath})\,d\hat{z} = \frac{E^{\frac{1}{2}}}{2}\frac{\epsilon_W}{\epsilon_T}\left[\frac{\tau^{(y)}}{f}\hat{\imath} - \frac{\tau^{(x)}}{f}\hat{\jmath}\right], \tag{3.5}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors parallel to the x and y axes respectively.

4. The thermocline region

In the region below the Ekman layer and removed from any side wall boundary-layer region, the interior equations, with the neglect of the viscous and heat diffusion terms, become (assuming $\sigma S \gg E$)

$$fv_I = p_{Ix}, \quad fu_I = -p_{Iy}, \quad T_I = p_{Iz}, \quad w_I = 0, \tag{4.1a, b, c, d}$$

$$u_{Ix} + v_{Iy} + w_{Iz} = 0. \tag{4.1e}$$

Elimination of the pressure yields the Sverdrup relation

$$\beta v_I = f w_{Iz} \tag{4.2}$$

and the 'thermal' wind balances

$$fu_{Iz} = -T_{Iy}, \quad fv_{Iz} = T_{Ix}. \tag{4.3}$$

However, (4.1d) and (4.2) imply that $T_{Ix} = 0$. This is in general incompatible with (3.4). This difficulty is resolved only through the introduction of a thermal boundary layer or thermocline. In fact it is not difficult to show also that no motion can occur beneath the thermocline layer at all. Even u_I must vanish if proper allowance is made for the condition that u is zero on $x = 0, 1$ in conjunction with the insulating condition $T_x = 0$ on $x = 0, 1$. This strong constraint arises solely due to the β-effect (4.2) which shows that northward motion on the sphere must be accompanied by vortex stretching which in the case of a stratified ocean is suppressed by the stable stratification. The required associated vertical velocity can occur only when there is sufficient thermal diffusion, as in the thermal boundary layer.

In the thermocline region the interior variables can then be shown to be represented in the following manner

$$u_I = (E/\sigma S)^{\frac{1}{4}}\, u_T(x, y, \zeta), \tag{4.4a}$$

$$v_I = (E/\sigma S)^{\frac{1}{4}}\, v_T(x, y, \zeta), \tag{4.4b}$$

$$w_I = (E/\sigma S)^{\frac{1}{2}}\, w_T(x, y, \zeta), \tag{4.4c}$$

$$p_I = (E/\sigma S)^{\frac{1}{4}}\, p_T(x, y, \zeta), \tag{4.4d}$$

$$T_I = T_T(x, y, \zeta), \tag{4.4e}$$

where
$$\zeta = (1-z)(\sigma S/E)^{\frac{1}{4}}.$$

The thermocline variables must vanish as ζ becomes large.

The dynamical equations for the thermocline variables are

$$fv_T = p_{Tx}, \quad fu_T = -p_{Ty}, \quad T_T = -p_{T\zeta}, \tag{4.5a, b, c}$$

$$u_{Tx} + v_{Ty} = w_{T\zeta}, \quad w_T = \tfrac{1}{2} T_{T\zeta\zeta}. \tag{4.5d, e}$$

If all variables are eliminated in favour of the temperature, we obtain the thermocline equation

$$T_{T\zeta\zeta\zeta\zeta} = 2\beta/f^2 T_{Tx}. \tag{4.6}$$

From (4.6) we can observe that no thermocline layer would exist if β were zero. When β is zero, no thermal layer is possible, but in that case none is needed for the Sverdrup constraint (4.2) is also absent.

The solution to the thermocline equation will satisfy the boundary conditions (3.3) and (4.4) and will limit the motions produced by the surface heating and wind stress to a region near the surface whose characteristic depth, l_T, is given by†

$$l_T = (E/\sigma S)^{\frac{1}{4}}$$

which is the same depth scale found by Stommel & Veronis (1959). Since no motion occurs beneath this depth no effect of the bottom, either frictional or topographical is present in this model.

Before proceeding to the solution of (4.6) it is convenient to Fourier transform (4.5) and (4.6). Let

$$\begin{Bmatrix} u_T \\ v_T \\ p_T \end{Bmatrix} = \frac{2}{\pi}\int_0^\infty \cos k\zeta \begin{Bmatrix} U_T(x, y, k) \\ V_T(x, y, k) \\ P_T(x, y, k) \end{Bmatrix} dk, \tag{4.7a}$$

$$\begin{Bmatrix} w_T \\ T_T \end{Bmatrix} = \frac{2}{\pi}\int_0^\infty \sin k\zeta \begin{Bmatrix} W_T(x, y, k) \\ \Theta_T(x, y, k) \end{Bmatrix}. \tag{4.7b}$$

† In dimensionless units, with the dependence on β_* explicit, the thermocline depth L_T is

$$L_T = \left[\kappa_V f_0^2/\beta_* g\alpha \frac{(\Delta T_V)}{D}\right]^{\frac{1}{4}} L^{\frac{1}{4}}.$$

The transformed variables then satisfy the equations

$$fV_T = P_{Tx}, \quad fU_T = -P_{Ty}, \quad \Theta_T/k = P_T, \qquad (4.8a, b, c)$$

$$U_{Tx} + V_{Ty} = kW_T - \frac{1}{2}\frac{\epsilon_W}{\epsilon_T}(\sigma S)^{\frac{1}{2}}\hat{k}.\operatorname{curl}\frac{\boldsymbol{\tau}}{f}, \quad W_T = -\tfrac{1}{2}k^2\Theta_T + \tfrac{1}{2}k\mathscr{T}(x,y). \qquad (4.8d, e)$$

The governing equation for Θ_T which is the transform of (4.6) is

$$\Theta_{Tx} - \frac{k^4 f^2}{2\beta}\Theta_T = \frac{f^2}{2\beta}\left[k(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\hat{k}.\operatorname{curl}\frac{\boldsymbol{\tau}}{f} - k^3\mathscr{T}\right], \qquad (4.9)$$

whose general solution is

$$\Theta_T = C(k,y)\exp\left[-(k^4f^2/2\beta)(1-x)\right]$$
$$+\int_x^1 \frac{f^2}{2\beta}\left[k^3\mathscr{T} - k(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\hat{k}.\operatorname{curl}\frac{\boldsymbol{\tau}}{f}\right]\exp\left[-\tfrac{1}{2}k^4f^2(x'-x)/\beta\right]dx'. \qquad (4.10)$$

Note that the general solution for Θ_T includes a free solution

$$C(k,y)\exp\left[-\tfrac{1}{2}k^4f^2(1-x)/\beta\right],$$

which can only be determined by an analysis of the side wall boundary layers at the ocean basin's rim. One of the central difficulties in completely specifying the thermocline solution is that the specification of the velocity u_T on $x = 1$ is not sufficient to determine the circulation in the free solution. This is quite different from a homogeneous ocean model (e.g. Pedlosky 1968) where it *is* sufficient. Specification of u_T on $x = 1$ will determine C_y but not C, and will therefore still leave unspecified an essential component of the circulation. This unspecified circulation will have no vertical mean, and it is not surprising that this difficulty does not arise in homogeneous models where the interior horizontal velocity is constrained by the Taylor–Proudman theorem to be depth independent.

Since

$$\begin{Bmatrix} U_T \\ V_T \\ P_T \end{Bmatrix} = \int_0^\infty \cos k\zeta \begin{Bmatrix} u_T \\ v_T \\ p_T \end{Bmatrix} d\zeta,$$

the vertical means of u_T, v_T and p_T are given by the values of their transforms evaluated at $k = 0$. Hence if only C_y is known the as yet indeterminate circulation will give no contribution to U_T and V_T when $k = 0$.

Further discussion of the thermocline solution and its complete determination must await the results of boundary-layer analysis on the side walls. To analyse the boundary layers on $x = 0$ and $x = 1$ it will be convenient to assume that $\tau^{(y)}(0,y) = \tau^{(y)}(1,y) = 0$, so that there is no stress-forced upwelling at these coasts. It is not difficult to remove this restriction but it does lead to greater complexity in presentation.

5. The meridional boundary layers

Consider the boundary-layer region near $x = 0$, i.e. the western boundary layer. The vertical scale of this region, since it must match on to the thermocline will also be l_T. Let the boundary-layer corrections to the interior variables be $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{T}$ and $\tilde{p}$. Assuming that the northward motion remains geostrophic and the whole motion remains hydrostatic the boundary-layer equations are, for the correction functions,

$$f\tilde{v} = \tilde{p}_x,$$
$$f\tilde{u} = -\tilde{p}_y + \tfrac{1}{2}E\tilde{v}_{xx},$$
$$0 = l_T^{-1}\tilde{p}_\zeta + \tilde{T},$$
$$\sigma S\tilde{w} = \tfrac{1}{2}E\tilde{T}_{xx},$$
$$\tilde{u}_x + \tilde{v}_y = \tilde{w}_\zeta;$$

or in terms of the temperature alone,

$$E\tilde{T}_{xxx} + (E/\sigma S)^{\frac{1}{2}} f^2 \tilde{T}_{x\zeta\zeta} - 2\beta\tilde{T} = 0. \tag{5.1}$$

If the boundary-layer variables are Fourier transformed in the same way as the thermocline variables, (5.1) becomes

$$E\tilde{\Theta}_{xxx} - k^2(E/\sigma S)^{\frac{1}{2}} f^2 \tilde{\Theta}_x - 2\beta\tilde{\Theta} = 0, \tag{5.2}$$

where $\tilde{\Theta}$ is the Fourier sine transform in ζ of $\tilde{T}$. If $k = O(1)$ then the resulting boundary-layer structure depends on the relation between σS and E. If $\sigma S \gg E^{\frac{1}{3}}$ then (5.2) can be approximated by neglecting the second term, yielding a boundary-layer thickness of $O(E^{\frac{1}{3}})$. The thickness and horizontal structure is then identical to the layer found by Munk (1950). The ζ (or k) dependence is carried parametrically and determined by matching to the thermocline solution. On the other hand, if $\sigma S \ll E^{\frac{1}{3}}$ the layer splits into two layers. The thicker layer has a thickness $O(l_T^2)$ and is a balance between the second and third terms while the thinner layer has a thickness $O(l_T(\sigma S)^{\frac{1}{2}})$ and is a balance between the first two terms. This dependence of the structure on the stratification is interesting and it is worthwhile noting that the equation for $\tilde{\Theta}$ when $k = 0$, which is the equation which holds for the vertically averaged boundary-layer variables, *always* yields a thickness of $O(E^{\frac{1}{3}})$ and a structure of a Munk layer. In a sense then, the equations for the mean can yield a boundary-layer structure which is never observed locally (i.e. for a particular value of ζ or k) in the stratified model unless the value of the stratification is sufficiently large. Since it does correspond to the mean structure, and occurs in the case of substantial stratification (which is the condition of interest here) the following analysis will be restricted to the former case, i.e. when $\sigma S \gg E^{\frac{1}{3}}$. One might also argue, although much less strenuously, that this is also justifiable in terms of the sizes of the parameters in cases of oceanic relevance. For if

$$\alpha(\Delta T_V) = 10^{-3}, \qquad g = 10\ \text{cm}^2\,\text{sec}^{-1},$$
$$D = 4\times10^5\ \text{cm}, \qquad \sigma = 10,$$
$$\nu_V = 10\ \text{cm}^2\,\text{sec}^{-1}, \quad f_0 = 10^{-4}\ \text{sec}^{-1},$$

then $\sigma S = 10^{-2}$ while $E = 0{\cdot}6\times10^{-6}$. But it is clear that σS is really of $O(E^{\frac{1}{3}})$ and the lack of definite information concerning the proper value of σ and ν_V makes the decision to consider $\sigma S \gg E^{\frac{1}{3}}$ really rather arbitrary. It is, however, a choice that is made.

Restricting ourselves then to circumstances where $\sigma S \gg E^{\frac{1}{3}}$ we can represent the dynamic variables in the western boundary-layer region as

$$u = l_T u_T + l_T \tilde{u}(\eta, y, \zeta), \tag{5.3a}$$

$$v = l_T v_T + l_T E^{-\frac{1}{3}} \tilde{v}(\eta, y, \zeta), \tag{5.3b}$$

$$w = l_T^2 w_T + E^{\frac{1}{3}} (\sigma S)^{-1} \tilde{w}(\eta, y, \zeta), \tag{5.3c}$$

$$p = l_T p_T + l_T \tilde{p}(\eta, y, \zeta), \tag{5.3d}$$

$$T = T_T + \tilde{T}(\eta, y, \zeta), \tag{5.3e}$$

where the correction functions go to zero as

$$\eta = x E^{-\frac{1}{3}}$$

becomes large. The solutions for the correction functions can easily be found and are

$$\tilde{v} = A(y, \zeta) \exp[-(2\beta)^{\frac{1}{3}} \tfrac{1}{2}\eta] \sin (2\beta)^{\frac{1}{3}} \tfrac{1}{2}\sqrt{3}\, \eta, \tag{5.4a}$$

$$\tilde{u} = -A_y (2\beta)^{-\frac{1}{3}} \exp[-(2\beta)^{\frac{1}{3}} \tfrac{1}{2}\eta] \sin [(2\beta)^{\frac{1}{3}} \tfrac{1}{2}\sqrt{3}\, \eta + \tfrac{1}{3}\pi], \tag{5.4b}$$

$$\tilde{T} = -A_\zeta f (2\beta)^{-\frac{1}{3}} \exp[-(2\beta)^{\frac{1}{3}} \tfrac{1}{2}\eta] \sin [(2\beta)^{\frac{1}{3}} \tfrac{1}{2}\sqrt{3}\, \eta + \tfrac{1}{3}\pi], \tag{5.4c}$$

$$\tilde{W} = A_\zeta \tfrac{1}{2} f (2\beta)^{\frac{1}{3}} \exp[-(2\beta)^{\frac{1}{3}} \tfrac{1}{2}\eta] \sin [(2\beta)^{\frac{1}{3}} \tfrac{1}{2}\sqrt{3}\, \eta + \tfrac{2}{3}\pi]. \tag{5.4d}$$

The scaling for the amplitude of the correction functions has been chosen so that this Munk layer can bring the zonal velocity u, to rest on $x = 0$. The form of the solution has already been arranged to satisfy the no slip condition on v to lowest order; at the same time this satisfies the insulating condition $T_x = 0$ on $x = 0$.

Interior to this layer, there exist thin non-hydrostatic and non-geostrophic regions which serve primarily only to satisfy the no slip condition on w without carrying significant flux of fluid. The thickness of this region is less than $E^{\frac{1}{3}}$ and will not be explicitly considered which means that *the no slip condition for w on* the lateral boundaries will be relaxed and this will not affect the determination of the interior fields or the significant thicker boundary layers to lowest order. It is still crucial however to satisfy the vertical mass flux balance, of which more later. For the moment, it is enough to point out that the vertical velocity in the $E^{\frac{1}{3}}$ layer is too small to have the vertical mass flux in this layer play any role in completing the vertical mass flux balance.

On $x = 1$, as examination of (5.2) shows, no solution (for any value of σS) can be found which at once satisfies the no slip condition on v and at the same time has sufficient freedom left in the form of the solution to satisfy the condition of $u = 0$ on $x = 1$ for an arbitrary u_T. This circumstance occurs also in the homogeneous models.

The boundary-layer corrections on $x = 1$ serve to satisfy the no slip condition on v, and by the thermal wind relation, automatically to satify the insulating condition $T_x = 0$. In fact the amplitude of the correction functions must be reduced by $E^{\frac{1}{3}}$ so that the thermocline zonal velocity itself must satisfy the condition $u = 0$ on $x = 1$, i.e. from (4.10), (4.7*c*) and (4.7*b*) we see that

$$C_y = 0,$$

so that C is a function of k alone. The free solution then implies that a circulation with the form

$$V_T = \tfrac{1}{2}(k^3 f/\beta)\, C(k) \exp\,[-k^4 f^2(1-x)/2\beta],$$

$$U_T = k^3(1-x)\, C(k) \exp\,[-k^4 f^2(1-x)/2\beta],$$

$$W_T = -\tfrac{1}{2}k^2 C(k) \exp\,[-k^4 f^2(1-x)/2\beta],$$

is yet left undetermined. If $k^3 C(k)$ goes to zero as $k \to 0$ then the associated horizontal velocities have zero mean. To determine $C(k)$ finally and thus to specify completely the interior circulation it is necessary to consider carefully the boundary layers on $y = 0$, b and to describe the closure of the vertical mass flux circuit by considering the upwelling phenomena in these boundary layers.

6. The northern boundary layer

Consider the boundary-layer region near $y = b$. The vertical scale of this region will also be, for the most part l_T. In the same manner as the discussion of the western boundary layer, let the boundary-layer corrections to the interior variables be denoted $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{T}$ and $\tilde{p}$. Assuming that $\tilde{u}$, the 'downstream' velocity remains geostrophic, and the motion is hydrostatic the boundary-layer equations for the correction functions are

$$0 = -\tilde{p}_x + f\tilde{v} + \tfrac{1}{2}E\tilde{u}_{yy} + \tfrac{1}{2}(E/l_T^2)\,\tilde{u}_{\zeta\zeta},$$

$$0 = -\tilde{p}_y - f\tilde{u},$$

$$0 = l_T^{-1}\tilde{p}_\zeta + \tilde{T},$$

$$\sigma S \tilde{w} = \tfrac{1}{2}E\tilde{T}_{yy} + \tfrac{1}{2}(E/l_T^2)\,\tilde{T}_{\zeta\zeta},$$

$$\tilde{u}_x + \tilde{v}_y = l_T^{-1}\tilde{w}_\zeta.$$

The ζ derivatives have been retained in the diffusion terms along with the y derivatives because the layer thickness of the northern boundary layer is as large as the thermocline depth l_T. This thickening of the northern region compared to the western region is due to the reduction of the advection of planetary vorticity (the β effect) due to the decrease in $\tilde{v}$ which is in this layer the velocity normal to the boundary.

Eliminating all variables in favour of the temperature yields the boundary-layer equation

$$E\tilde{T}_{yyyy} + (E/\sigma S)^{\frac{1}{2}} f^2 \tilde{T}_{yy\zeta\zeta} + f^2 \tilde{T}_{\zeta\zeta\zeta\zeta} - 2\beta\tilde{T}_x = 0, \tag{6.1}$$

where use has been made of the fact that $\sigma S \ll 1$.

Both the third and fourth terms in (6.1) are $O(1)$ (this is in fact the thermocline balance) and the structure of the layer depends on which of the first two terms enter the balance. If the first term balances the third and fourth the layer thickness is $E^{\frac{1}{4}}$, but this requires that $\sigma S \gg 1$ to ignore the second term. This is not very realistic, as noted in §5, and the alternative possibility must be considered. If $\sigma S \ll 1$ then the layer splits into two. The thicker layer is a balance between the second, third and fourth terms and yields a boundary-layer thickness l_T which is the thermocline thickness. This layer is as thick as it is deep (in the scaled co-ordinates) which is why the ζ derivatives were retained along with y derivatives. Interior to this region lies a thinner region of thickness $l_T(\sigma S)^{\frac{1}{2}}$, in which the first two terms of (6.1) balance. They merge and become the $E^{\frac{1}{4}}$ layer as σS approaches unity.

Again, it is convenient to introduce the appropriate Fourier transforms of the dynamic variables, viz.

$$\left.\begin{aligned} \begin{Bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{p} \end{Bmatrix} &= \frac{2}{\pi}\int_0^\infty \cos k\zeta \begin{Bmatrix} \tilde{U} \\ \tilde{V} \\ \tilde{P} \end{Bmatrix} dk, \\ \begin{Bmatrix} \tilde{w} \\ \tilde{T} \end{Bmatrix} &= \frac{2}{\pi}\int_0^\infty \sin k\zeta \begin{Bmatrix} \tilde{W} \\ \tilde{\Theta} \end{Bmatrix} dk. \end{aligned}\right\} \tag{6.2}$$

In transforming (6.1) information is required at $\zeta = 0$. In particular the problem demands the knowledge of the vertical velocity pumped into the boundary layer from the upper Ekman layer. As long as $\sigma S \ll 1$ it is possible to show that, as in the homogeneous model (Pedlosky 1968) the Ekman layer transport impinging on the boundary at $y = b$ spews out of an $E^{\frac{1}{2}}$ by $E^{\frac{1}{2}}$ corner at $y = b$, $z = 1$ and enters the side wall boundary layer only in the region $b - y \leqslant O(E^{\frac{1}{2}})$. In fact, as long as $\sigma S \ll 1$ the analysis of the corner proceeds entirely as in the homogeneous case. This implies that on $\zeta = 0$ (Pedlosky 1968),

$$\tilde{w}(x,y,0) = \frac{\partial}{\partial y}[\exp[-\lambda(b-y)]\{U_E(x,b)\sin\lambda(b-y) - V_E(x,b)\cos\lambda(b-y)\}], \tag{6.3}$$

where $\lambda = f^{\frac{1}{2}}E^{-\frac{1}{2}}$. This wind-stress forced upwelling must be taken into account in the solution of (6.1). By taking the sine transform of (6.1) and using the boundary condition (6.3) we find that $\tilde{\Theta}$ satisfies

$$\begin{aligned} E\tilde{\Theta}_{yyyy} - k^2 l_T^2 f^2 \tilde{\Theta}_{yy} + k^4 f^2 \tilde{\Theta} - 2\beta\Theta_x \\ = -\frac{2f^2k^2\lambda}{l_T^2}\{\exp[-\lambda(b-y)][(U_E(x,b) - V_E(x,b)\sin\lambda(b-y) \\ -(U_E(x,b) - V_E(x,b))\cos\lambda(b-y)]\}. \end{aligned} \tag{6.4}$$

A particular solution of (6.4) can easily be found in the form

$$\tilde{\Theta}_p = \exp[-\lambda(b-y)][A\sin\lambda(b-y) + B\cos\lambda(b-y)],$$

$$\text{where} \quad A = \frac{2\lambda k}{l_T^2}\left\{\frac{(V_E(x,b) - U_E(x,b))(k^4 - 4E^{-1}) - (V_E + U_E)2f(\sigma sE)^{-\frac{1}{2}}k^2}{k^8 + 4k^4f^2/\sigma SE + 16/E^2}\right\}, \tag{6.5}$$

$$B = \frac{2\lambda k}{l_T^2}\left\{\frac{(V_E + U_E)(k^4 - 4E^{-1}) - (V_E - U_E)2f(\sigma SE)^{-\frac{1}{2}}k^2}{k^8 + 4k^4f^2/\sigma SE + 16/E^2}\right\}, \tag{6.6}$$

where again use has been made of the fact that $\sigma S \ll 1$, $E^{\frac{1}{2}} \ll 1$. Within the same level of approximation the associated northward velocity in the particular solution, $\tilde{V}_p$, can be found and on $y = b$ it has the form,

$$\tilde{V}_p(x, b, k) = \frac{V_E(x, b)\, 16/E^2 - 2k^2 U_E(x, b)\,[k^4/l_T^3 + 4f^2 l_T/E^2]}{l_T(k^8 + 4k^4 f^2/\sigma SE + 16/E^2)}. \tag{6.7}$$

Now the limit $k \to 0$ yields $\quad l_T \tilde{V}_p(x, b, 0) = V_E(x, b)$

and this is just the integrated $\tilde{v}$ in the particular solution, i.e.

$$l_T \int_0^\infty \tilde{v}_p(x, b, \zeta)\, d\zeta = l_T \tilde{V}_p(x, b, 0) = V_E(x, b). \tag{6.8}$$

Therefore on $y = b$ the combined northward velocity of the thermocline solution and the homogeneous solution of (6.4) must accept a total horizontal flux of mass equal to $-V_E(x, b)$. It is convenient to think of the particular solution as a separate physical region in which the flux in the upper Ekman layer descends and enters the deeper oceanic circulation below. As far as the homogeneous solution of (6.4) is concerned the source flow at the top is replaced by a source of fluid entering into the region horizontally on $y = b$. The distribution of this flow with depth is of importance. Thus on $y = b$,

$$\tilde{v}_p(x, b, \zeta) = \frac{2}{\pi} \int_0^\infty \cos k\zeta\, \tilde{V}_p(x, b, k)\, dk. \tag{6.9}$$

Examination of (6.7) shows that the integral in (6.9) can be easily evaluated in terms of the poles of (6.7) which are at

$$\left.\begin{aligned} k = k_{1n} &= 2^{\frac{1}{2}}(\sigma S/Ef^2)^{\frac{1}{4}}\{\exp[i(\tfrac{1}{4}\pi + \tfrac{1}{2}n\pi)]\} \\ k = k_{2n} &= 2^{\frac{1}{2}}(f^2/\sigma SE)\{\exp[i(\tfrac{1}{4}\pi + \tfrac{1}{2}n\pi)]\} \end{aligned}\right\} \quad (n = 0, 1, 2, 3). \tag{6.10}$$

The contributions to (6.9) from the first set of poles is, for $\sigma S \ll 1$, negligible compared to the second set. In fact for $\sigma S \ll 1$ a consistent approximation to (6.7) is

$$\tilde{V}_p(x, b, k) = \frac{V_E}{l_T}\left\{\frac{4\sigma S/Ef^2}{k^4 + 4\sigma S/Ef^2}\right\} - \frac{2k^2 U_E}{l_T^4}\left\{\frac{1}{k^4 + 4\sigma S/Ef^2}\right\}. \tag{6.11}$$

If, for example, the wind stress is parallel to the line $y = b$, so that $U_E = 0$, then $\tilde{v}_p(x, b, \zeta)$ is simply

$$\tilde{v}_p(x, b, \zeta) = \frac{V_E(x, b)\sqrt{2}}{l_T f^{\frac{1}{2}}}\{\exp[-\zeta/l_T f^{\frac{1}{2}}]\sin(\zeta/l_T f^{\frac{1}{2}} + \tfrac{1}{4}\pi)\}. \tag{6.12}$$

The important thing to note is that this particular solution decays with depth and is vanishing small when $1 - z = O(l_T^2)$! Thus the penetration of the particular solution is very shallow† compared to the thermocline depth and this will have important consequences in what follows.

The homogeneous solution of (6.4) can now be found. When $\sigma S \ll 1$ it decomposes, as already described, into two boundary-layer regions. In the thicker

† In dimensional units the penetration depth, L_P, is

$$L_P = \left\{\kappa_H f_0/g\alpha \frac{(\Delta T)_V}{D}\right\}^{\frac{1}{2}}.$$

layer, with scale l_T, the boundary-layer correction functions can be represented as follows:

$$\tilde{V} = l_T \tilde{V}_1(x, r, k), \tag{6.13a}$$

$$\tilde{U} = \tilde{U}_1(x, r, k), \tag{6.13b}$$

$$\tilde{P} = l_T \tilde{P}_1(x, r, k), \tag{6.13c}$$

$$\tilde{W} = l_T^2 \tilde{W}_1(x, r, k), \tag{6.13d}$$

$$\tilde{\Theta} = \tilde{\Theta}_1(x, r, k), \tag{6.13e}$$

where
$$r = (b-y)\, l_T^{-1}.$$

These correction functions satisfy the dynamical equations

$$0 = -\tilde{P}_{1x} + f\tilde{V}_1, \tag{6.14a}$$

$$0 = \tilde{P}_{1r} - f\tilde{U}_1, \tag{6.14b}$$

$$0 = k\tilde{P}_1 + \tilde{\Theta}_1, \tag{6.14c}$$

$$\tilde{W} = \tfrac{1}{2}[\tilde{\Theta}_{1rr} - k^2\tilde{\Theta}_1], \tag{6.14d}$$

$$\tilde{U}_{1x} - \tilde{V}_{1r} = l_T k \tilde{W}_1. \tag{6.14e}$$

Eliminating all variables in favour of $\tilde{U}_1$

$$\tilde{U}_{1rr} + \frac{2\beta}{f^2k^2}\tilde{U}_{1x} - k^2\tilde{U}_1 = 0. \tag{6.15}$$

Before proceeding to the solution of (6.15) it is important to note that (6.14e) implies that the motion within this region represented by the boundary-layer correction variables is horizontally non-divergent to $O(l_T)$. Thus there is no significant *vertical* flux of mass in this outer layer. Fluid entering this region, either from the interior, or the source region represented by the particular solution will flow essentially horizontally.

Using the fact that $\tilde{U}_1$ must be zero on $x = 1$ (for the same reason the thermocline zonal velocity must vanish there) (6.15) can be solved by a Laplace transform in $(1-x)$ to yield

$$\tilde{U}_1 = \frac{\exp[-f^2/2\beta k^4(1-x)]}{2\pi i}\int_{-\infty i}^{\infty i} F(\alpha)\exp\left[-r\left(\frac{2\beta}{fk^2}\right)^{\frac{1}{2}}\alpha^{\frac{1}{2}} + \alpha(1-x)\right]d\alpha, \tag{6.16a}$$

$$\tilde{V}_1 = \frac{\exp[-f^2/2\beta k^4(1-x)]}{2\pi i}\int_{-\infty i}^{\infty i} F(\alpha)\left(\frac{fk^2}{2\beta}\right)^{\frac{1}{2}}\frac{\alpha - f^2k^4/2\beta}{\alpha^{\frac{1}{2}}}$$
$$\times \exp\left[-r\left(\frac{2\beta}{fk^2}\right)^{\frac{1}{2}}\alpha^{\frac{1}{2}} + \alpha(1-x)\right]d\alpha. \tag{6.16b}$$

As one can see, either from the formal solutions (6.16) or from the original equation (6.15) this outer layer has some of the characteristics of a diffusion layer, with the layer broadening as $x = 0$ is approached from the east.

In the region $b-y = O(l_T(\sigma S)^{\frac{1}{2}})$ additional correction functions must be added to the solution of (6.4), these are denoted by a subscript 2. In this region it can be shown that

$$\tilde{V} = l_T \tilde{V}_1(x,r,\zeta) + l_T(\sigma S)^{\frac{1}{2}} \tilde{V}_2(x,\mu,\zeta), \tag{6.17a}$$

$$\tilde{U} = \tilde{U}_1(x,r,\zeta) + \tilde{U}_2(x,\mu,\zeta), \tag{6.17b}$$

$$\tilde{W} = l_T^2 \tilde{W}_1(x,r,\zeta) + E^{\frac{1}{2}}(\sigma S)^{-1} \tilde{W}_2(x,\mu,\zeta), \tag{6.17c}$$

$$\tilde{\Theta} = \tilde{\Theta}_1(x,r,\zeta) + (\sigma S)^{\frac{1}{2}} \tilde{\Theta}_2(x,\mu,\zeta), \tag{6.17d}$$

where $$\mu = (b-y)(l_T^2 \sigma S)^{-\frac{1}{2}}.$$

From the scaling amplitudes in (6.17), two important observations can be made. First, only the northward velocity in the outer layer will enter into the lowest order matching of v on $y = b$ while the inner layer serves only to satisfy the no slip condition on u. Although this latter condition is important it will not affect the determination of either the interior nor any other boundary-layer flow to lowest order. It is, in distinction to the outer layer, passive, and does not directly figure in any further matching. For brevity, therefore, it will not be considered further in detail. Secondly, this layer has a vertical velocity which, as in the case of the outer layer, has an amplitude too small to provide a vertical flux of the same order as the interior. In fact, a detailed consideration of this inner layer shows that it too is also horizontally non-divergent.

In summary then, the northern boundary layer can be considered broken up into three parts. In the outer layer the northward velocity in the interior is brought to rest and turned westward; in the next layer the eastward velocity is brought to rest, while the innermost region represented by the particular solution is the only one of the three regions in which a significant amount of vertical mass flux occurs, and this only to a very shallow depth.

The remaining calculation to be performed in the northern boundary layer is the matching of v, or its transform, i.e. on $y = b$

$$V_T + \tilde{V}_1 + \tilde{V}_p/l_T = 0, \tag{6.18a}$$

or using (4.10), (4.7), (6.16b) and (6.11):

on $y = b$

$$C(k)\frac{k^3 f}{2\beta}\exp\left[-k^4 \tfrac{1}{2} f^2(1-x)/\beta\right] + \frac{f}{2\beta}\left[(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\hat{k}\,.\,\mathrm{curl}\,\boldsymbol{\tau}/f - k^2\mathscr{T}\right]$$
$$+\frac{k^3 f^3}{4\beta^2}\int_x^1 \left[k^3\mathscr{T} - k(\sigma S)^{\frac{1}{2}}\hat{k}\,.\,\mathrm{curl}\,\boldsymbol{\tau}/f\right]\exp\left[-k^4\tfrac{1}{2}f^2(x'-x)/\beta\right]dx'$$
$$+\frac{V_E}{l_T}\left[\frac{4\sigma S/Ef^2}{k^4 + 4\sigma S/Ef^2}\right] - \frac{2k^2 U_E}{l_T^4}\left[\frac{1}{k^4+4\sigma S/Ef^2}\right]$$
$$+\frac{\exp\left[-\frac{1}{2}f^2k^4(1-x)/\beta\right]}{2\pi i}\int_{-\infty i}^{\infty i} F(\alpha)\exp\left[\alpha(1-x)\right]\frac{\alpha - f^2k^4/2\beta}{\alpha^{\frac{1}{2}}}\,d\alpha = 0. \tag{6.18b}$$

Once $\boldsymbol{\tau}$ and $\mathscr{T}$ have been specified as a function of x on $y = b$; $F(\alpha)$ can be found. For example, if $\boldsymbol{\tau}$ is a function only of y and directed parallel to the x axis, while $\mathscr{T}$ is also a function only of y, it is easy to show that

$$\tilde{V}_1 = \exp\left[-\tfrac{1}{2}f^2k^4(1-x)/\beta\right]\left[C\frac{k^3f}{2\beta}-\frac{f}{2\beta}\left\{k^2\mathscr{T}-(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\hat{k}.\operatorname{curl}\boldsymbol{\tau}/f\right\}\right]$$
$$\times E_{rfc}\left(\left[\frac{\beta}{2fk^2}\right]^{\frac{1}{2}}\frac{n}{(1-x)^{\frac{1}{2}}}\right)-\frac{16(\sigma S)^{\frac{1}{2}}V_E E^{-\frac{1}{2}}}{E^2\left\{\dfrac{4k^4f^2}{\sigma SE}+\dfrac{16}{E^2}\right\}}\left[\begin{aligned}&e^{rk}E_{rfc}\left(\left[\frac{\beta}{2fk^2}\right]^{\frac{1}{2}}\frac{r}{(1-x)^{\frac{1}{2}}}+fk^2\left(\frac{1-x}{2\beta}\right)^{\frac{1}{2}}\right)\\&+e^{-rk}E_{rfc}\left(\left[\frac{\beta}{2fk^2}\right]^{\frac{1}{2}}\frac{r}{(1-x)^{\frac{1}{2}}}-fk^2\left(\frac{1-x}{2\beta}\right)^{\frac{1}{2}}\right)\end{aligned}\right]. \tag{6.19}$$

The analysis of the boundary-layer region near $y = 0$ proceeds precisely as it did for the region near $y = b$. The results of that computation therefore, will merely be presented as needed.

7. Closure of the thermocline problem

It is still necessary to determine the free thermocline solution, i.e. to determine the function $C(k)$ in (4.10). This can most easily be done by utilizing the obvious global constraint that

$$\int_0^b\int_0^1 dx\,dy\,w = 0.$$

The net upwelling in the oceanic interior must be balanced by the downwelling in the boundary layers on the oceanic rim. Now we can see why it was so important to note that the downwelling in the side-wall layers was so shallow compared to the thermocline depth. Below this penetration depth of $O(l_T^2)$ the thermocline vertical velocity must have zero area average. To determine $C(k)$ then, we have

$$l_T^2\int_0^b\int_0^1 dx\,dy\,W_T = -\int_0^b\int_0^1 dx\,dy\,\tilde{W}_p, \tag{7.1}$$

where $\tilde{W}_p$ refers to the downwelling velocities in the boundary layers on $y = 0, b$. Again, for simplicity let us restrict our attention to the case where U_E is zero on $y = 0, b$. This is not essential, and in fact the net downwelling is due to V_E. If, however, $U_E = 0$ on $y = 0, b$, then from (6.11) and the continuity equation we have:

$$\int_0^b\int_0^1 dx\,dy\,\tilde{W}_p = -\int_0^1 V_E(x,b)\,dx\,\frac{k^3}{k^4+4\sigma S/Ef^2}+\int_0^1 V_E(x,0)\,dx\,\frac{k^3}{k^4+4\sigma S/Ef^2}. \tag{7.2}$$

Since
$$W_T = -\tfrac{1}{2}k^2\Theta_T+k\mathscr{T}, \tag{7.3}$$

then
$$\int_0^b\int_0^1 dx\,dy\,W_T = -\tfrac{1}{2}k^2\int_0^b\int_0^1 dx\,dy\,\Theta_T. \tag{7.4}$$

Finally (7.1), (7.2), (7.4) and (4.10) imply that

$$
\begin{aligned}
C(k)\int_0^b dy[1-\exp(-f^2/2\beta)]\frac{2\beta}{f^2} \\
= -\int_0^b dy\int_0^1 dx\int_x^1 dx'\frac{f^2}{2\beta}\left[k^3\mathcal{T}-k(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\hat{k}.\operatorname{curl}\boldsymbol{\tau}/f\right]\exp\left[-\frac{k^4f^2}{2\beta}(x'-x)\right] \\
-(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\int_0^1 dx\left[\frac{k^3}{k^4+4\sigma S/Ef^2}\right]\frac{\tau^{(x)}(x,b)}{2f(b)} \\
+(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\int_0^1 dx\left[\frac{k^3}{k^4+4\sigma S/Ef^2}\right]\frac{\tau^{(x)}(0,b)}{2f(0)}. \qquad (7.5)
\end{aligned}
$$

The determination of $C(k)$ is then complete and the interior thermocline solution completely specified. For k of $O(1)$ the last two terms of (7.5) are negligible. This implies, in turn, that for $(1-z) = O(l_T)$ the net interior vertical mass flux must vanish. Only when the surface is approached will the net interior vertical mass flux differ from zero. Inverting (7.2) ,we find that

$$
\begin{aligned}
\int_0^b\int_0^1 dx\,dy\,\tilde{w}_p &= -\int_0^b\int_0^1 dx\,dy\,w_T l_T^2 \\
&= \frac{\epsilon_W}{\epsilon_T}E^{\frac{1}{2}}\left[\begin{array}{l}\int_0^1 dx\frac{\tau^{(x)}(x,b)}{2f(b)}\exp[-\zeta(l_T^2f(b))^{-\frac{1}{2}}]\cos\zeta(l_T^2f(b))^{-\frac{1}{2}} \\ -\int_0^1 dx\frac{\tau^{(x)}(0,b)}{2f(0)}\exp[-\zeta(l_T^2f(0))^{-\frac{1}{2}}]\cos\zeta(l_T^2f(0))^{-\frac{1}{2}}\end{array}\right]. \qquad (7.6)
\end{aligned}
$$

The interior vertical mass flux has zero areal average below a depth of order $l_T^2 = (E/\sigma S)^{\frac{1}{2}}$. On $\zeta = 0$, (7.6) yields

$$
l_T^2\int_0^b\int_0^1 dx\,dy\,w_T = \frac{\epsilon_W}{\epsilon_T}E^{\frac{1}{2}}\oint\frac{\boldsymbol{\tau}}{2f}.d\mathbf{l}, \qquad (7.7)
$$

where $d\mathbf{l}$ measures distance around the rim of the basin. The right-hand side of (7.7) is the area average of the Ekman flux (3.3) and (7.7) shows that the net flux penetrates only to $(1-z) = O(l_T^2)$. Of course there are significant vertical velocities in the thermocline to depths of $O(l_T)$ but they have no average in a horizontal plane.

8. Example of an x independent forcing

Consider the case where the wind stress acts purely in the zonal direction and is independent of longitude, i.e. $\boldsymbol{\tau} = \tau^{(x)}(y)\hat{\imath}$ while the surface temperature is also longitude independent, i.e. $\mathcal{T} = \mathcal{T}(y)$, then (4.10) yields

$$
\begin{aligned}
T_T = \frac{2}{\pi}\int_0^\infty dk\sin k\zeta\Big[C(k)\exp[-\tfrac{1}{2}k^4f^2(1-x)/\beta]+\left(k(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\frac{d}{dy}\frac{\tau^{(x)}}{f}+k^3\mathcal{T}\right) \\
\times\left(\frac{1-\exp[-\frac{1}{2}k^4f^2(1-x)/\beta]}{k^4}\right)\Big], \qquad (8.1)
\end{aligned}
$$

$$
\begin{aligned}
v_T = \frac{2}{\pi}\int_0^\infty dk\cos k\zeta\Big[C(k)\frac{k^3f}{2\beta}\exp[-\tfrac{1}{2}k^4f^2(1-x)/\beta] \\
-\frac{f^2}{2\beta}\left((\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\frac{d}{dy}\frac{\tau^{(x)}}{f}+k^2\mathcal{T}\right)\exp[-\tfrac{1}{2}k^4f^2(1-x)/\beta]\Big], \qquad (8.2)
\end{aligned}
$$

where

$$C(k) = \int_0^b dy\left[k^3\mathscr{T} + k(\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_T}\frac{d}{dy}\frac{\tau^{(x)}}{f}\right][1-\exp\{\tfrac{1}{2}k^4f^2/\beta\}]/k^8f^2$$
$$-\frac{\epsilon_W}{\epsilon_T}\frac{(\sigma S)^{\frac{1}{2}}}{k^3}\left[\frac{\tau^{(x)}(b)}{f(b)}\frac{4\sigma S/Ef^2(0)}{k^4+4\sigma S/Ef^2(b)} - \frac{\tau^{(x)}(0)}{f(0)}\frac{4\sigma S/Ef^2(0)}{k^4+4\sigma S/Ef^2(0)}\right]. \quad (8.3)$$

Note that $C(0) = 0$, so that the horizontal circulation in the 'free' solution has zero vertical average. (In addition the vertical mean of the circulation produced by the heating is zero.) On $x = 1$, we see that

$$T_T = 2\pi^{-1}\int_0^\infty dk \sin k\zeta\, C(k).$$

The determination of $C(k)$ really then specifies the temperature distribution with depth on the eastern wall of the ocean. If $C(k)$ were zero the temperature departure from the original linear temperature gradient would be zero. To get some idea of the thermocline structure it is illuminating to examine in some detail the structure of a typical dynamical field, for example v_T.

For small ζ the structure of the solution will depend on the forcing but for large ζ we can expect that the intrinsic nature of the thermocline solution will reveal itself. In addition, for large ζ it is possible to evaluate (8.2) asymptotically, for example, by the method of steepest descent. The calculation is straight forward and yields

$$v_T \sim -\frac{2^{\frac{1}{3}}(\sigma S)^{\frac{1}{2}}}{f(3\pi)^{\frac{1}{2}}}\left(\frac{f^2}{\beta}\right)^{\frac{5}{6}}\frac{\epsilon_W}{\epsilon_T}\frac{d}{dy}\frac{\tau^{(x)}}{f}\frac{1}{(1-bx)^{\frac{1}{6}}\zeta^{\frac{1}{3}}}$$
$$\times\left[\exp\left\{-\frac{3}{8}\left(\frac{\beta}{2f^2}\right)^{\frac{1}{3}}\left(\frac{\zeta^4}{1-x}\right)^{\frac{1}{3}}\right\}\cos\left(\frac{3\sqrt{3}}{8}\left(\frac{\beta}{2f^2}\right)^{\frac{1}{3}}\left(\frac{\zeta^4}{1-x}\right)^{\frac{1}{3}}-\frac{\pi}{6}\right)\right]$$
$$-\frac{2^{\frac{3}{2}}}{f(3\pi)^{\frac{1}{2}}}\left(\frac{f^2}{2\beta}\right)^{\frac{1}{6}}\mathscr{T}(y)\frac{\zeta^{\frac{1}{3}}}{(1-x)^{\frac{1}{6}}}$$
$$\times\left[\exp\left\{-\frac{3}{8}\left(\frac{\beta}{2f^2}\right)^{\frac{1}{3}}\left(\frac{\zeta^4}{1-x}\right)^{\frac{1}{3}}\right\}\cos\left(\frac{3\sqrt{3}}{8}\left(\frac{\beta}{2f^2}\right)^{\frac{1}{3}}\left(\frac{\zeta^4}{1-x}\right)^{\frac{1}{3}}+\frac{\pi}{6}\right)\right]$$
$$+\frac{2^{-\frac{3}{2}}}{f(3\pi)^{\frac{1}{2}}}\left(\frac{\beta}{f^2}\frac{\zeta}{1-x}\right)^{\frac{7}{6}}2\,\mathrm{Re}\,C\left(\left[\frac{\beta}{2f^2}\frac{\zeta}{(1-x)}\right]^{\frac{1}{3}}\right)$$
$$\times\left[\exp\left\{\left(\frac{\beta}{2f^2}\right)^{\frac{1}{3}}\left(\frac{\zeta^4}{1-x}\right)^{\frac{1}{3}}\left(-\frac{3}{8}+\frac{i3\sqrt{3}}{8}\right)+\frac{\pi i}{3}\right\}\right], \quad (8.4)$$

(where Re denotes the real part).

The deep structure of thermocline is readily apparent. For large ζ the rapid variation of the solution depends on the function in the curly brackets in (8.4). This function is constant along lines where $\zeta^4(1-x) = \text{constant}$. Thus the isolines of constant v_T shallow as $x = 1$ is approached from the west. If $C(k)$ were zero then each isoline of constant v_T would rise to the surface at $x = 1$. This would happen, for example, if the wind stress were zero and if βb were sufficiently small so that f could be treated as a constant in (8.3).

Finally, to complete the solution, it is necessary to determine the strength of the western boundary current, i.e. to determine $A(y,\zeta)$ in (5.4a) (or equivalently, its cosine transform $\mathscr{A}(y,\zeta)$).

On $x = 0$, $u_T + \tilde{u}$ must vanish, or from (5.4b) and (4.7b, c)

$$\frac{3^{\frac{1}{2}}}{2}(2\beta)^{-\frac{1}{3}}\frac{d}{dy}\mathscr{A}(y,k) = (fk)^{-1}\frac{\partial}{\partial y}\Theta_T(0,y,k), \tag{8.5}$$

thus
$$\mathscr{A}(y,k) = -\frac{2}{\sqrt{3}}\frac{(2\beta)^{\frac{1}{3}}}{k}\left[\int_y^b \frac{1}{f}\frac{\partial\Theta_T}{\partial y'}(0,y',k)\,dy' + K(k)\right]. \tag{8.6}$$

To determine $K(k)$, note that the intersection of the layer on the western wall with the l_T layer on the northern wall produces a region in which the $E^{\frac{1}{3}}$ layer, with its dynamics unchanged, acts as western boundary layer for an extended interior which includes the l_T layer. Then (8.5) should be in fact altered to

$$\frac{\sqrt{3}}{2}(2\beta)^{-\frac{1}{3}}\frac{d}{dy}\mathscr{A}(y,k) = (fk)^{-1}\frac{\partial\Theta_T}{\partial y}(0,y,k) - \frac{\tilde{U}_1}{l_T}(0,r,k) \tag{8.7}$$

in regions near $y = b$. If (8.7) is integrated in y and use is made of the fact that southward velocity entering the western boundary layer in the l_T by $E^{\frac{1}{3}}$ corner in the northwest corner can come only from the l_T boundary-layer mass flux, we obtain

$$\frac{\sqrt{3}}{2}(2\beta)^{-\frac{1}{3}}\mathscr{A}(y,k) = -\int_y^b (fk)^{-1}\frac{\partial\Theta_T}{\partial y'}(0,y',k)\,dy' + \int_y^b \frac{\tilde{U}_1}{l_T}\left(0,\frac{b-y'}{l_T},k\right)dy', \tag{8.8}$$

then for all y such that $(b-y) \gg l_T$

$$\mathscr{A}(y,k) = \frac{2}{\sqrt{3}}(2\beta)^{\frac{1}{3}}\left[-\int_y^b (fk)^{-1}\frac{\partial\Theta_T}{\partial y'}(0,y',k)\,dy' + \int_0^\infty \tilde{U}_1(0,n,k)\,dn\right], \tag{8.9}$$

which in comparison with (8.6) determines the constant $K(k)$. By (6.14e) and the fact that $\tilde{U}_1 = 0$ on $x = 1$ we have

$$\int_0^\infty \tilde{U}_1(0,r,k)\,dr = \int_0^1 \tilde{V}_1(x,0,k)\,dx. \tag{8.10}$$

The matching condition (6.18a) that

$$\int_0^\infty \tilde{U}_1(0,r,k)\,dn = -\int_0^1 [\tilde{V}_p(x,0,k) + V_T(x,b,k)]\,dx \tag{8.11}$$

with (6.11), (4.7a) and (4.7b) yields,

$$\mathscr{A}(y,k) = \frac{2}{\sqrt{3}}(2\beta)^{\frac{1}{3}}\left[-\int_y^b (fk)^{-1}\frac{\partial\Theta_T}{\partial y'}(0,y',k)\,dy' - \frac{(\Theta_T(1,b,k) - \Theta_T(0,b,k))}{k} - \frac{V_E(b)}{l_T^2}\left(\frac{4\sigma S/Ef^2}{k^4 + 4\sigma S/Ef^2}\right)\right]. \tag{8.12}$$

As a check, we note that, using (4.10) and (3.5)

$$\mathscr{A}(y,0) = (\sigma S)^{\frac{1}{2}}\frac{\epsilon_W}{\epsilon_{T'}}(2\beta)^{-\frac{2}{3}}\frac{\partial\tau^{(x)}}{\partial y}. \tag{8.13}$$

This yields the vertically integrated northward flux in the western boundary current, and aside from factors introduced in the non-dimensionalization, agrees with the transport theories of the oceanic circulation.

The vertical averages of the western boundary layer are not surprising, but the vertical structure predicted by the theory is. Perusal of (8.12) shows that the western boundary layer has two vertical scales: the thermocline scale l_T forced by the interior transport impinging directly on the western wall; and another, shallower, component of the flow, which decays in a depth of $O(l_T^2)$, and is produced by the fluid which, downwelling at the northern boundary to a depth of $O(l_T^2)$ was swept to the western boundary in the northern l_T layer. If $V_E(b)$ is greater (less) than zero there is downwelling (upwelling) at the northern boundary and a shallow component of southward (northward) flow in the western boundary layer.

9. Conclusion

Even in this very simple physical model certain important interdependencies of more elementary aspects of the oceanic circulation have become apparent. It seems clear now that the complete understanding of the mid-ocean thermocline circulation cannot be completely divorced from a detailed consideration of the various boundary-layer phenomena at the ocean basin's rim, especially the problem of upwelling. In addition, as the example of the last section showed, even the structure of the western boundary layer is affected by boundary-layer processes which are not local.

As was stated in the introduction, the revelation of this interdependency of the elements of the circulation has been made possible only by the drastic mathematical and physical simplifications in the model. It is quite clear that the precise predictions of structure and amplitude given by this theory will be in serious error where non-linear effects are important. Nevertheless, this simple model, I believe, can serve as a guide in tracing similar effects in more physically realistic and complex models of the oceanic circulation.

Other problems within this framework would be interesting to study. In particular, the circulation produced when one of the side walls is heated or cooled, (especially the northern boundary) *below* the depth of the thermocline would be of great interest.

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